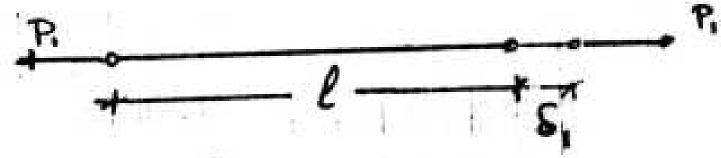
# Problemas Estáticamente indeterminados

Clase 13

Trabajo de las fuerzas interiores, Energía potencial de la deformación en la flexión

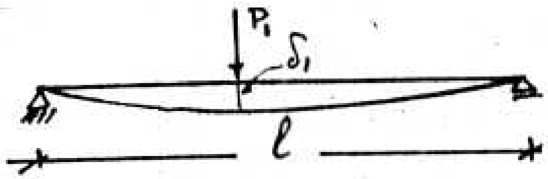
# Trabajo de las cargas exteriores - Ejemplos

## a) Barras cargadas axialmente

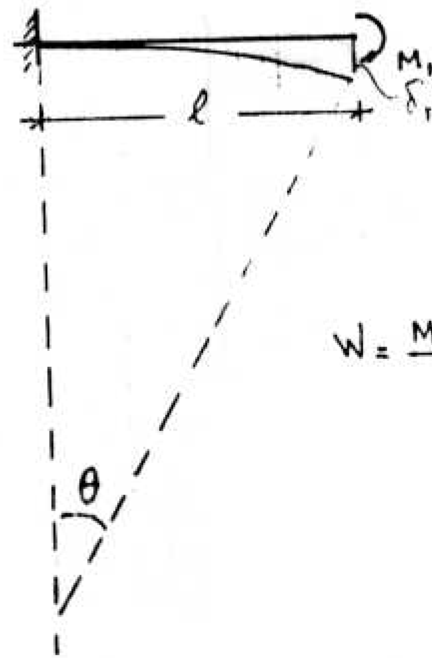


$$W = \frac{P\delta_1}{2}$$

## b) Barras flexionadas



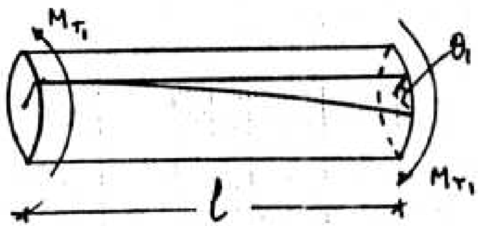
$$W = U = \frac{P_1\delta_1}{2}$$



$$W = \frac{M_1\theta}{2}$$

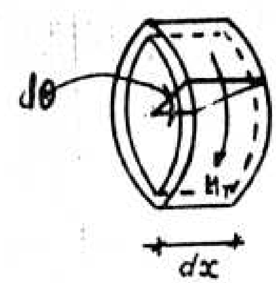
## c) Barras torsionadas

c.1. Barras cilíndricas circulares



$$W = \frac{M_{T_1}\theta}{2}$$

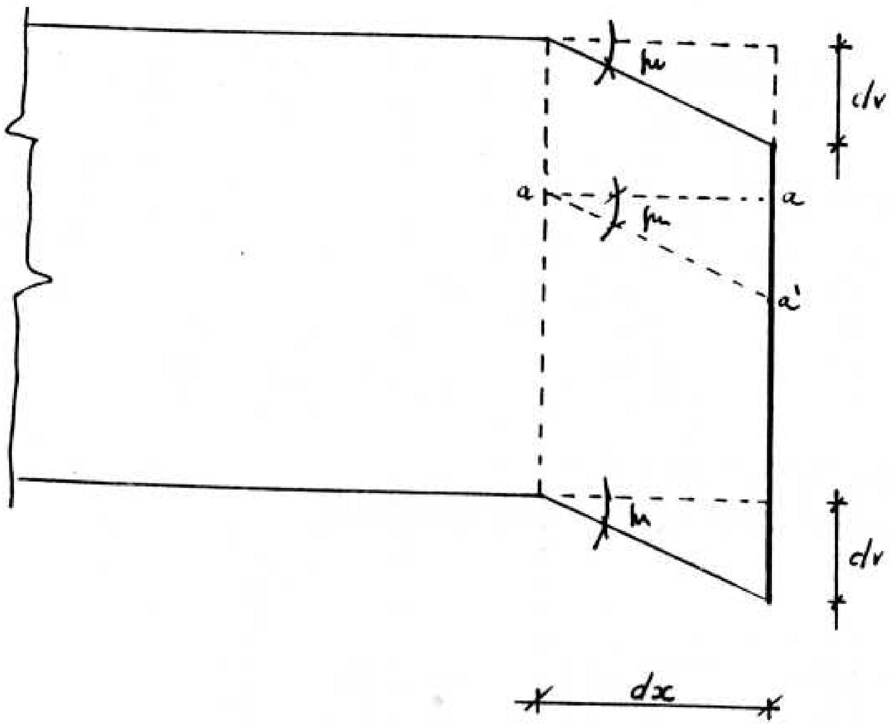
c.2. Barras de sección hueca de paredes delgadas (por unidad de longitud)



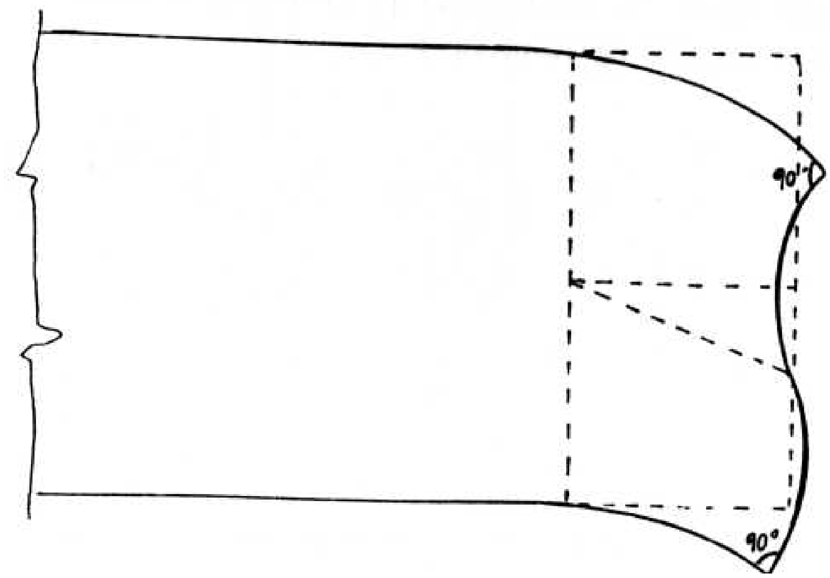
$$W = \frac{M_T}{2}\,\frac{d\theta}{dx}$$

# Tensiones Tangenciales

DEFORMACION SUPUESTA




DEFORMACION REAL



# Tensiones Tangenciales

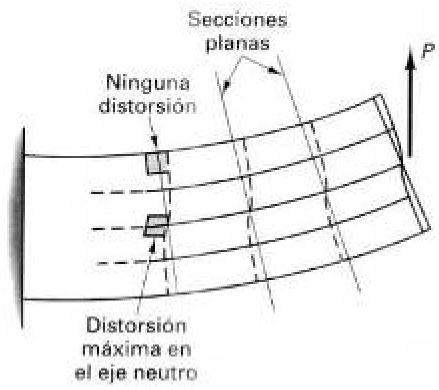



**Fig. 10-12** Distorsiones por cortante en una viga.

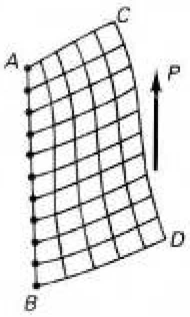



**Fig. 10-13** Red deformada para un voladizo corto de una solución por elemento finito.

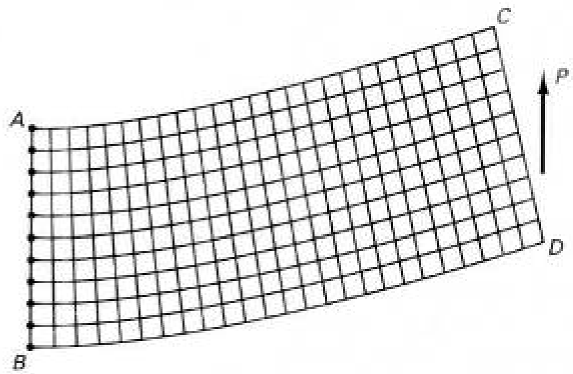
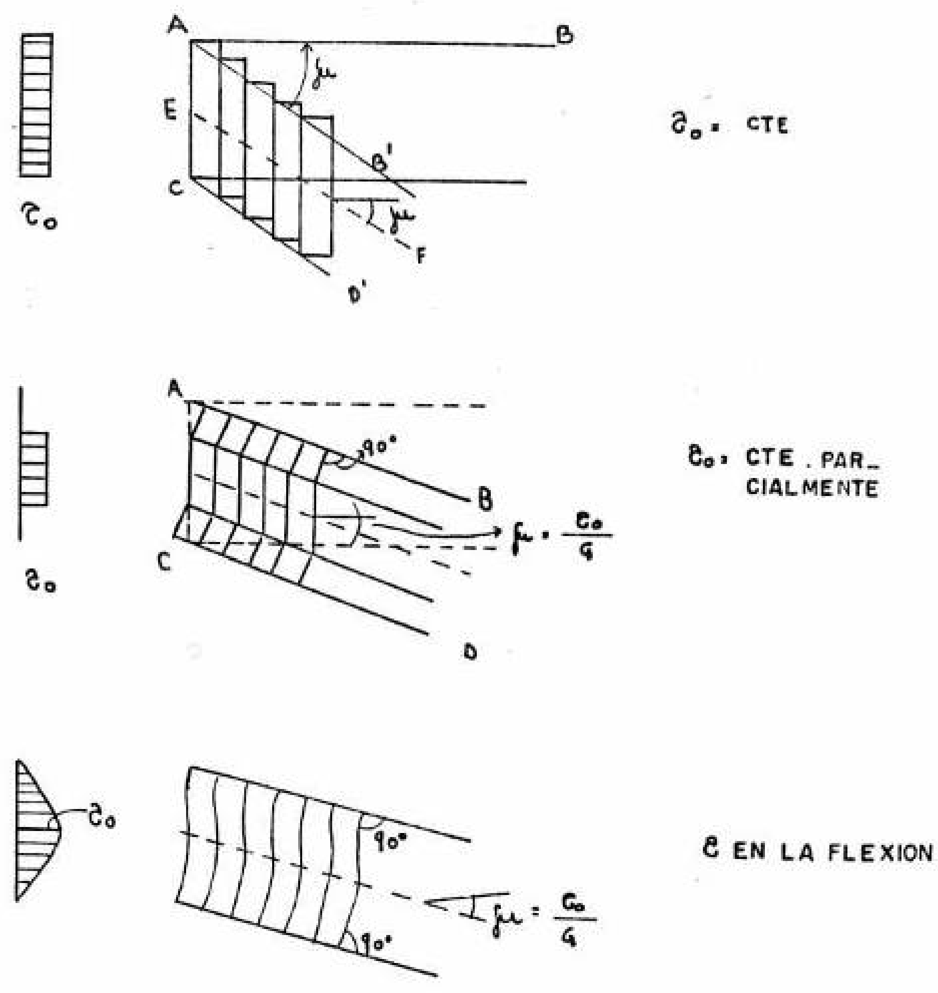


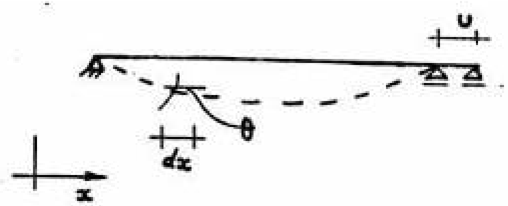

**Fig. 10-14** Solución por elemento finito que muestra la deformación de un voladizo moderadamente largo.

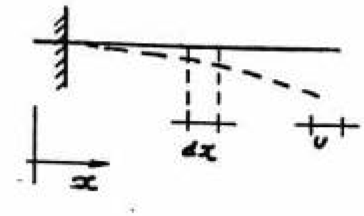
# Influencia de la fuerza cortante

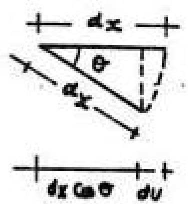



$$\gamma = \frac{dy_1}{dx} = \frac{\tau_o}{G} = \frac{k\,Q(x)}{A\,G} \qquad \frac{d^2y_1}{dx^2} = \frac{k}{AG}\,\frac{dQ(x)}{dx} = \frac{k\,q(x)}{A\,G}$$

# Desplazamiento horizontal




$$dx - du = dx \cdot \cos\theta$$

$$du = dx\,(1-\cos\theta) = dx \cdot \frac{\theta^2}{2}$$

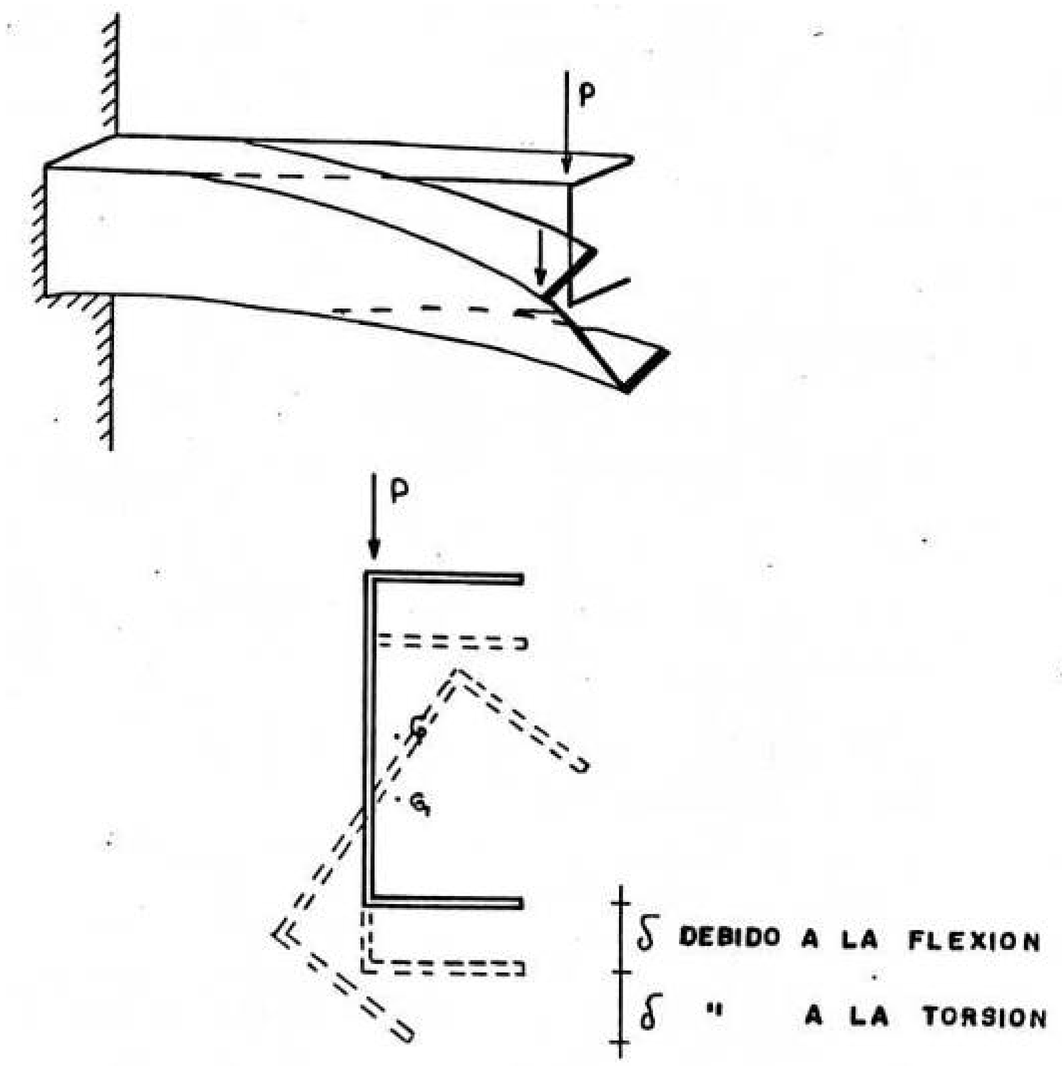
$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \frac{\theta^6}{6!} + \cdots$$

$$du = \frac{1}{2}\theta^2 dx = \frac{1}{2}\left(\frac{dy}{dx}\right)^2 dx$$

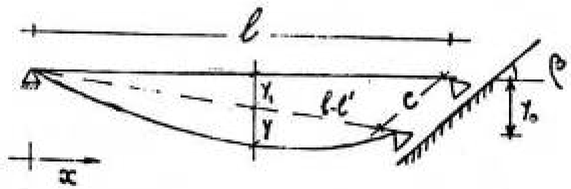
$$\frac{du}{dx} = \frac{1}{2}\left(\frac{dy}{dx}\right)^2$$

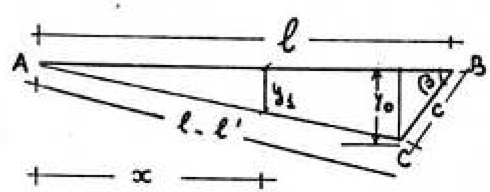
$$U = \int_0^{\ell} \frac{\theta^2}{2}\,dx = \int_0^{\ell} \frac{1}{2}\left(\frac{dy}{dx}\right)^2 dx$$

# Influencia de los momentos torsores



# Influencia inclinación de apoyos




$\underline{Y_0 = c \operatorname{sen} \beta}$

rlamiento: $\ell' = U + \Delta\ell$

$$\Delta\ell = \frac{N\ell}{AE}$$

$$U = \int_0^{\ell} \frac{1}{2}\left(\frac{dY}{dx}\right)^2 dx$$

$$(\ell - \ell')^2 = (\ell - c\cos\beta)^2 + y_0^2$$

$$(\ell - \ell')^2 = \ell^2 + c^2 - 2c\ell\cos\beta$$

si $\ell'' = \dfrac{\ell' - \ell'^2}{2\ell}$

$$\underline{c = \ell\cos\beta\left(1 - \sqrt{1 - \frac{2\ell''}{\ell\cos\beta}}\right)}$$

si $Y \simeq y_0$ $-$ $c^2 \simeq \ell'^2 \simeq 0$

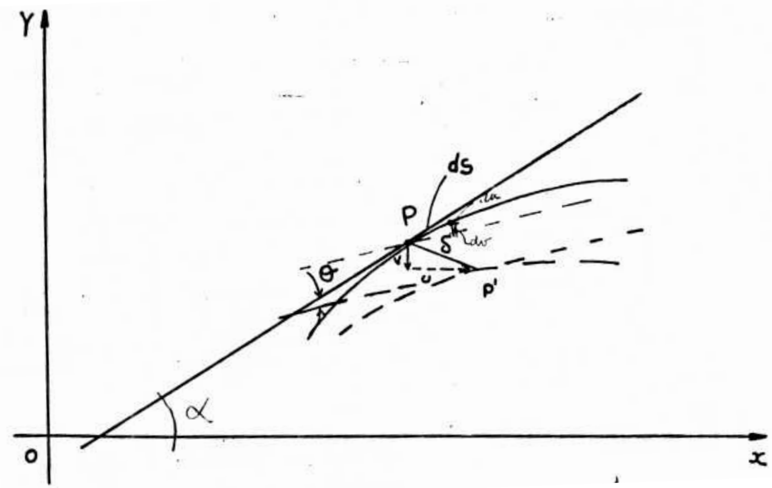
$$\ell^2 - 2\ell\ell' = \ell^2 - 2c\ell\cos\beta$$

$$c = \ell' \sec\beta$$

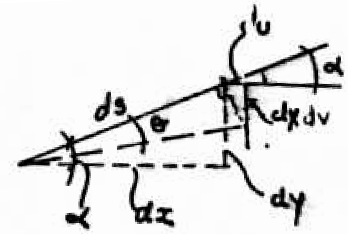
$$y_0 = \ell' \operatorname{tg}\beta$$

$$Y_{TOTAL} = Y + Y_1 = Y + Y_0\frac{x}{\ell} = Y + x\frac{\ell'}{\ell}\operatorname{tg}\beta$$

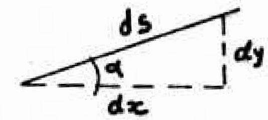
# Barras de pequeña curvatura




$$d\theta = -\frac{M}{EI}\,ds = -\frac{M}{EI}\,\frac{dx}{\cos\alpha} = -\frac{M}{EI}\,\frac{dy}{\operatorname{sen}\alpha}$$

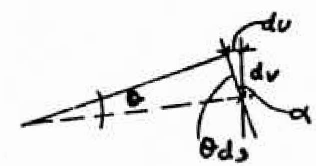



y $\quad \theta\, ds = \dfrac{du}{\operatorname{sen}\alpha} = \dfrac{dv}{\cos\alpha}$




$$\operatorname{sen}\alpha = \frac{dy}{ds} \;;\; \cos\alpha = \frac{dx}{ds}$$

$$\theta\, ds = \frac{du}{dy}\,ds = \frac{dv}{dx}\,ds$$

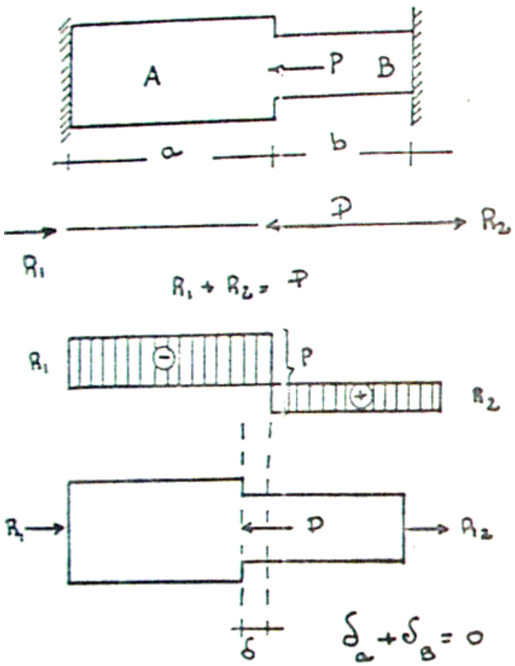



$$\theta = \frac{du}{dy} = \frac{dv}{dx}$$

$$\frac{d\theta}{dx} = \frac{d^2v}{dx^2} = -\frac{M}{EI\cos\alpha} \;;\; \frac{d\theta}{dy} = \frac{d^2u}{dy^2} = -\frac{M}{EI\operatorname{sen}\alpha}$$

# Método utilizando las relaciones de desplazamientos

## Cargas Normales



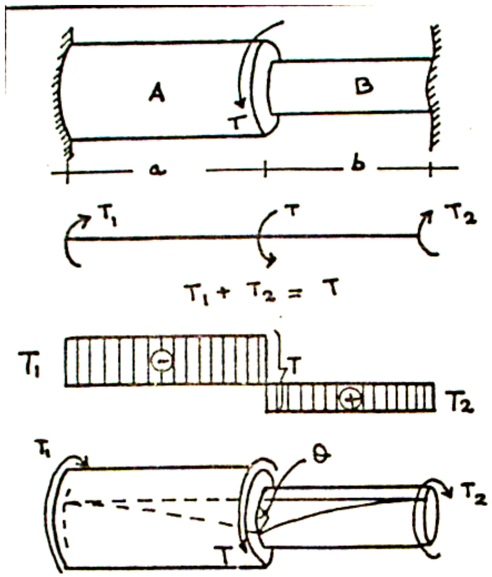

$R_1 + R_2 = P$

$\delta_a + \delta_B = 0$

$\delta_a = -\dfrac{R_1 a}{E_a A_a}$ $\qquad$ $\delta_B = \dfrac{R_2 b}{E_b A_b}$

$$R_2 = \frac{P}{1 + \frac{b}{a}\,\frac{E_a A_a}{E_b A_b}}$$

## Torsión



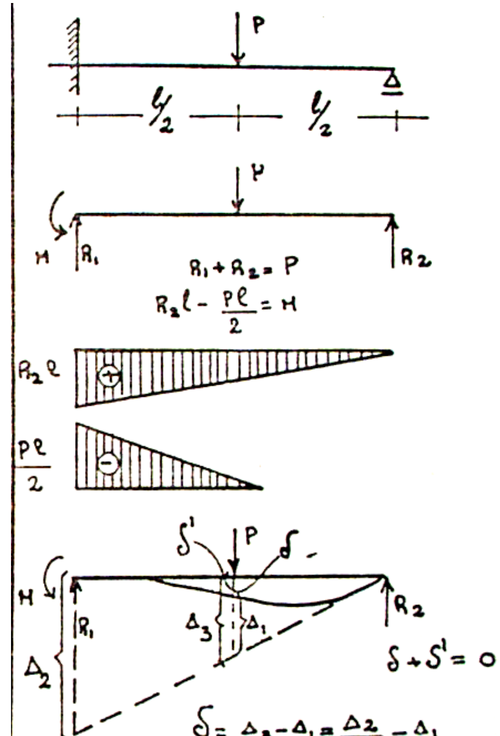

$T_1 + T_2 = T$

$\theta_a + \theta_b = 0$

$\theta_a = -\dfrac{T_1 a}{G_a I_{p_a}}$ $\qquad$ $\theta_b = \dfrac{T_2 b}{G_b I_{p_b}}$

$$\theta = \frac{T_1}{1 + \frac{b}{a}\,\frac{G_a I_{p_a}}{G_b I_{p_b}}}$$

## Flexión




$R_1 + R_2 = P$

$R_2 l - \dfrac{Pl}{2} = H$

$\delta + \delta' = 0$

$\delta = \Delta_3 - \Delta_1 = \dfrac{\Delta_2}{2} - \Delta_1$

$EI\,\dfrac{\Delta_2}{2} = \dfrac{R_2 l^3}{12} - \dfrac{Pl^3}{96}$ $\qquad$ $EI\,\Delta_1 = \dfrac{R_2 l^3}{48}$

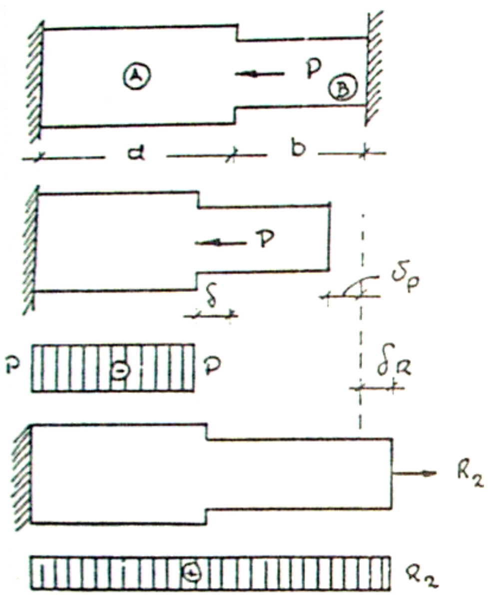
$$\left.\begin{aligned} EI\,\delta &= \frac{R_2 l^3}{16} - \frac{Pl^3}{96} \\ EI\,\delta' &= \frac{5R_2 l^3}{48} - \frac{Pl^3}{24} \end{aligned}\right\}$$

$R_2 = \dfrac{5}{16} P$

$R_1 = \dfrac{11}{16} P$

# Método de Superposición

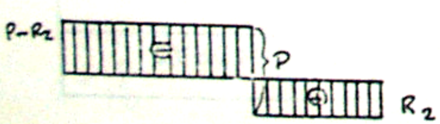
## Cargas Normales



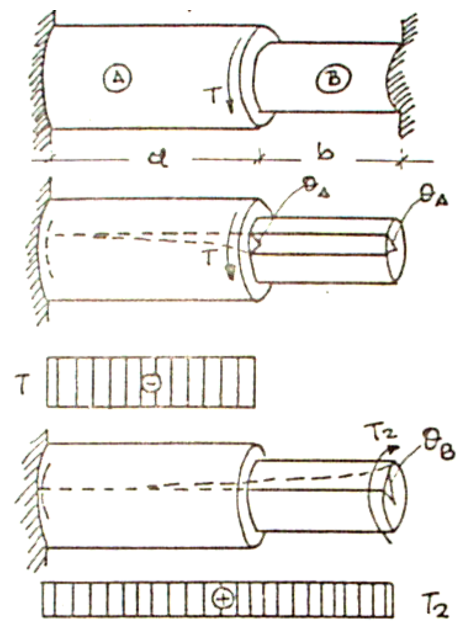
$$\delta_p + \delta_a = 0 \qquad R_1 + R_2 = P$$

$$-\frac{Pa}{E_a A_a} + \frac{R_2 b}{E_b A_b} + \frac{R_2 a}{E_a A_a} = 0$$

$$R_2 = \frac{P}{1 + \frac{b}{a}\,\frac{E_a A_a}{E_b A_b}}$$



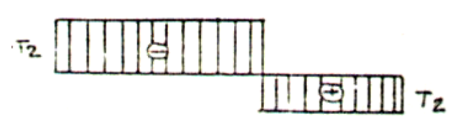
## Torsión



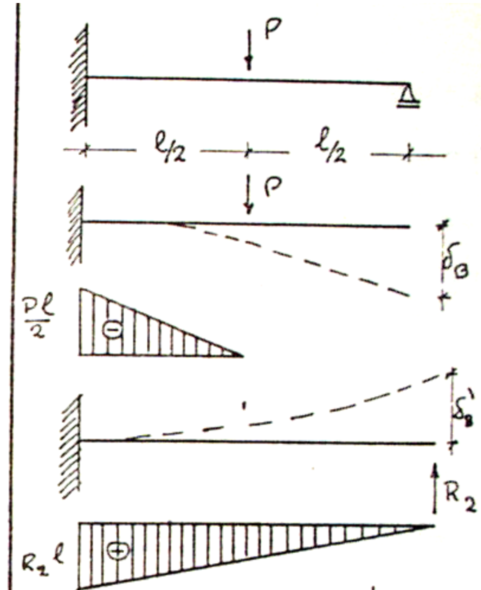
$$\theta_A + \theta_B = 0 \qquad T_1 + T_2 = T$$

$$-\frac{Ta}{G_a I_{pa}} + \frac{T_2 b}{G_b I_{pb}} + \frac{T_2 a}{G_a I_{pa}} = 0$$

$$T_2 = \frac{T}{1 + \frac{b}{a}\,\frac{G_a I_{pa}}{G_b I_{pb}}}$$



## Flexión



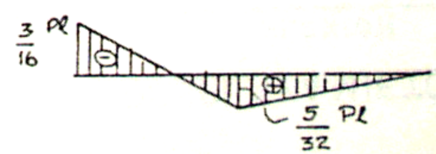
$$\delta_B + \delta'_B = 0$$

$$EI\,\delta_B = \frac{5}{48} P\ell^3$$

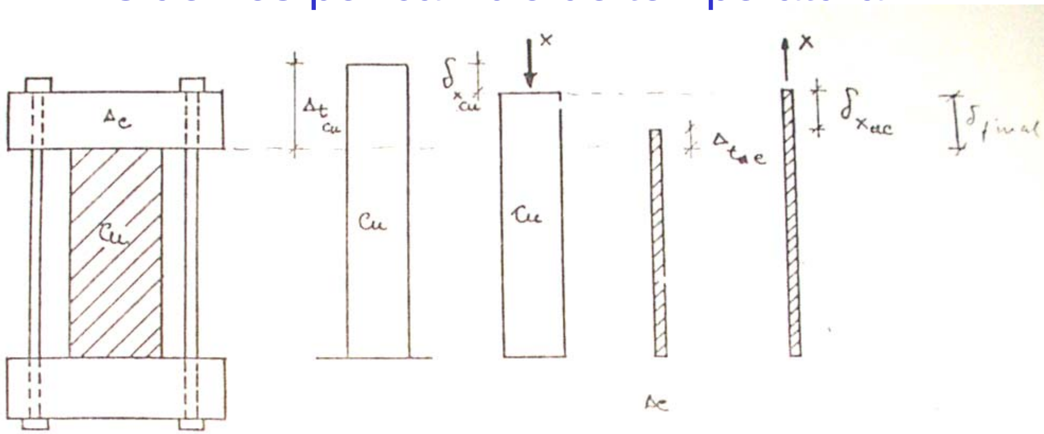
$$EI\,\delta_B = -\frac{R_B \ell^3}{3}$$

$$R_B = \frac{5}{16} P$$

$$R_A = \frac{11}{16} P$$



# Esfuerzos por cambio de temperatura

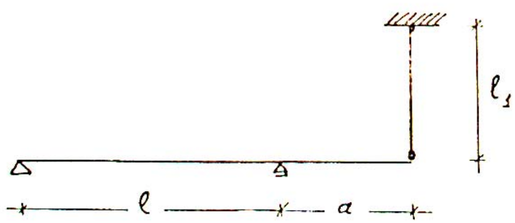


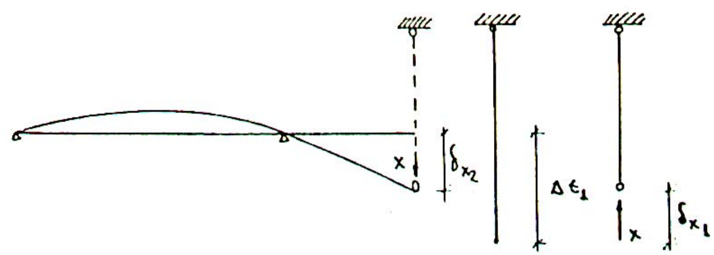
$\alpha_{cu} > \alpha_{ac}$

$\Delta_{t_{cu}} - \delta_{x_{cu}} = \Delta_{t_{ac}} + \delta_{x_{ac}}$

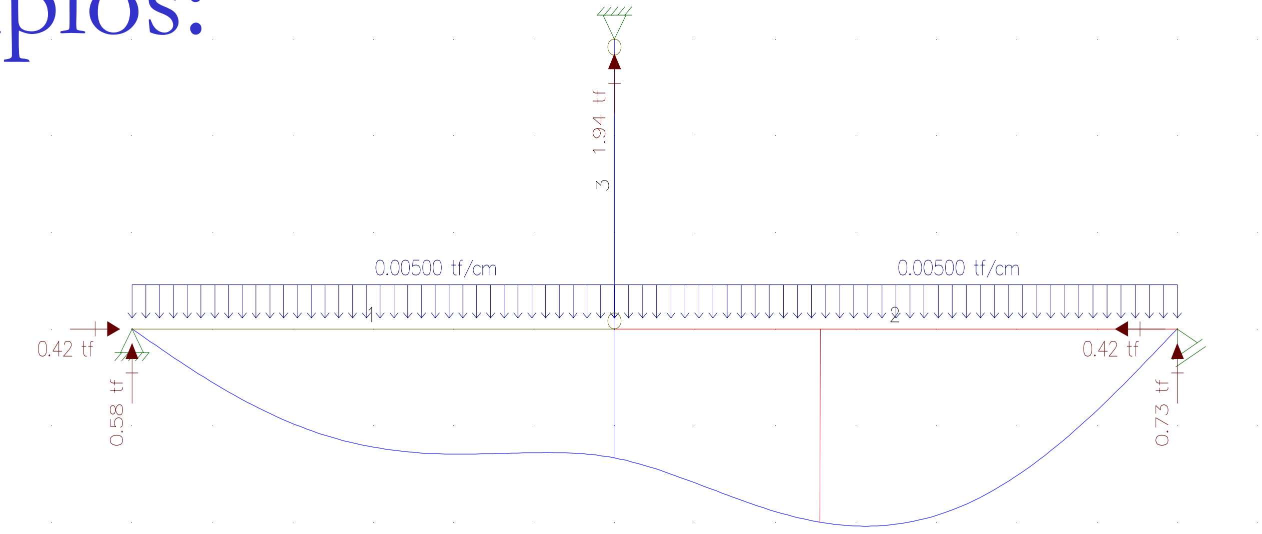
$\Delta t$

$\Delta_{t_1} = \delta_{x_1} + \delta_{x_2}$





# Ejemplos: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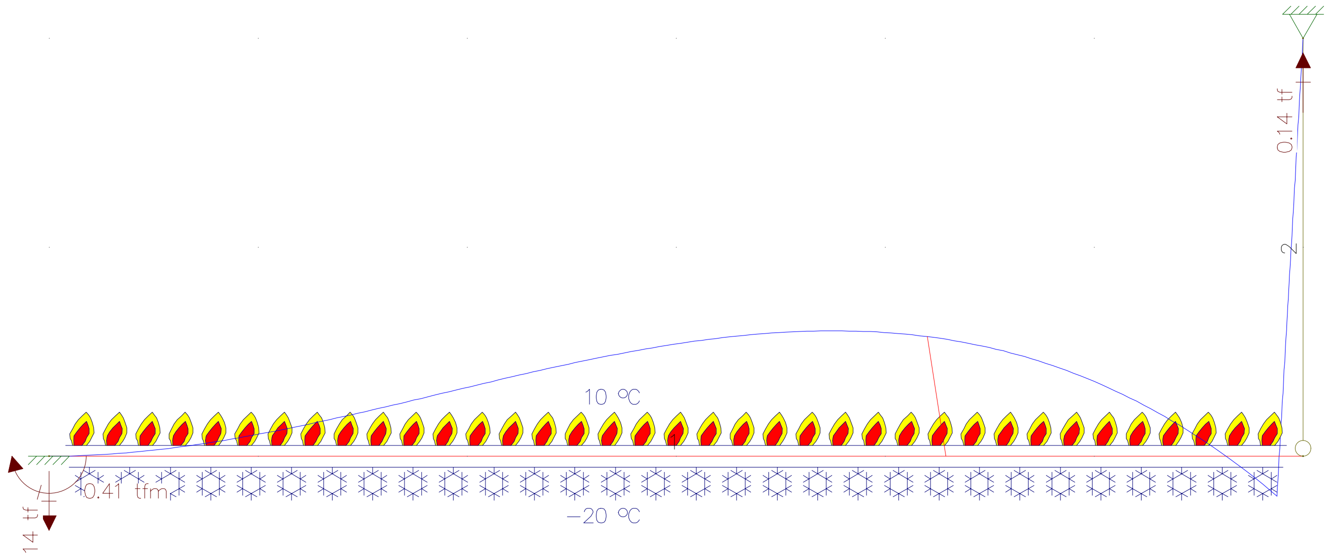
0.00500 tf/cm
0.00500 tf/cm
1
2
0.42 tf
0.42 tf
0.58 tf
0.73 tf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2
10 ºC
0.41 tfm
14 tf
−20 ºC

# Próxima Clase: Flexión Oblicua

Fin